Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

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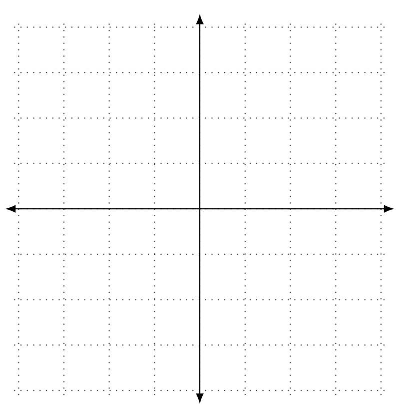
Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

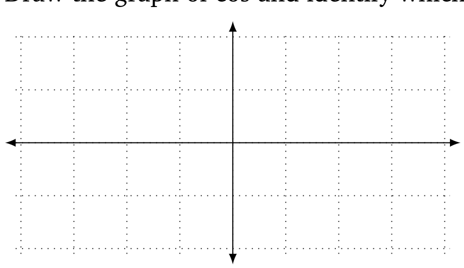
24.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.



24.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.



24.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

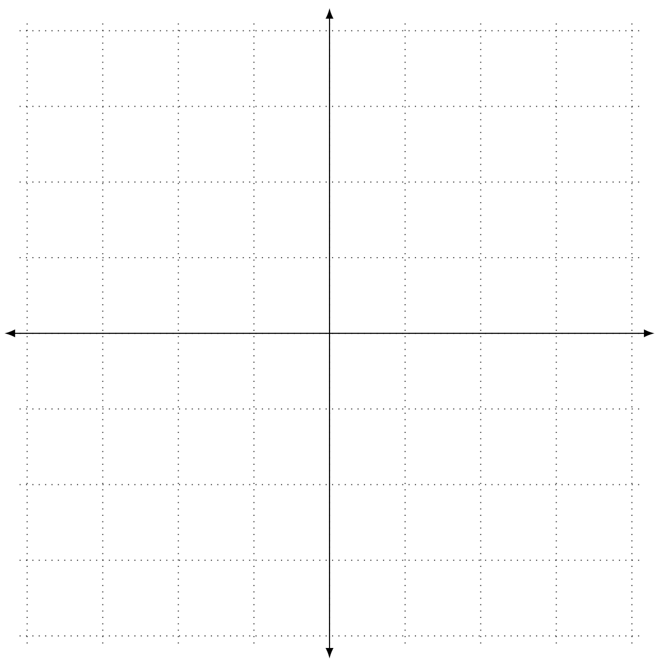
(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

24.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?



THM

For a vector $\vec{v} \in \mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.

DEF

Distance

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

DEF

Unit Vector

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

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25 Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

25.1 Find the distance between $\vec{u}$ and $\vec{v}$.

25.2 Find a unit vector in the direction of $\vec{u}$.

25.3 Does there exist a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

25.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal

DEF

Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v}=0$. The word orthogonal is synonymous with the word perpendicular.

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26.1 Find two vectors orthogonal to $\vec{a}=\begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

26.2 Find two vectors orthogonal to $\vec{b}=\begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

26.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\|=5$ and $\|\vec{y}\|=3$. What is the distance between $\vec{x}$ and $\vec{y}$?